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REGULAR PAPER

# Dynamically updating approximations based on multi-threshold tolerance relation in incomplete interval-valued decision information systems

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**Abstract**
With the development of society, data noise and other factors will cause the incompleteness of information systems. Objects may increase or decrease over time in information systems. The classical information system can be extended to the incomplete interval-valued decision information system (IIDIS) that is the researching object of this paper. Incremental learning technique is a significant method for solving approximate sets under dynamic data. This article defines a multi-threshold tolerance relation based on the set pair analysis theory and establishes a rough set model in IIDIS. Then, several methods and algorithms for statically/dynamically solving approximate sets are shown. Finally, comparative experiments from six UCI data sets show both dynamic algorithms take less time than the static algorithm to calculate the approximate sets no matter how object set changes.

**Keywords** Dynamic data · Multi-threshold tolerance relation · Approximation set · Incomplete interval-valued decision information system

## 1 Introduction

The advancement of computer networks provides human with a mass of information every day, including serviceable information and redundant information. The amount of information grows, so does the demand for information analysis tools. People expect to gain knowledge straightway from the data without any prior knowledge. Rough set theory (RST) [1–3] is such a mathematical tool raised by Pawlak [1]. Its primal archetype was derived from a relatively plain information model. Particularly in the past 30 years, the rough set theory has been largely applied in areas of science and engineering, for example, data mining and data

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analysis, approximate reasoning, decision support and machine learning algorithms, and so on [4–6]. Primary thought of RST is to achieve knowledge discovery by means of the classification of equivalence relations and to form concepts and regulations via the classification in an approximate space. What an equivalence class calculated from the equivalence relation is called concept is a set of indiscernible objects. Each uncertainty concept can be represented by a couple of accurate notions that are upper approximation and lower approximation.

An information system (IS) can be seen as a form for collecting knowledge and information. Rows represent the object set and columns show the attribute set (feature set). Usually the data we collect from real life are discrete, single-valued, or feature-valued (using language to describe attributes of objects). In many practical problems, the attribute values of information system are often continuous or only fall in an interval due to the complexity of practical problems, which extend the attribute values of the information system from real numbers to interval numbers. When the upper bound of interval number is equal to the lower bound, interval-valued information system degenerates into a classical information system. In the following, some scholars regard the interval-valued information system as their researching background of discussion. Both Dai et al. [7–9] and Zhang et al. [10] studied the metrics of information systems and provided some tools to promote the measurement of rough sets. Sun designed a dominance relation in intervals and proposed theorems for solving reductions by means of some diverting characteristics in Sun et al. [11]. The paper [12] has a distinctive view of information theory about reductions and thus generated attribute reduction algorithms. There is a special IS when attribute set is regarded as two parts, namely condition attribute set and decision attribute set. We call the IS a decision information system(DIS). Zhang [13] designed an algorithm and counted all reductions under considering the uncertainty of decision rules. In Qian et al. [14], in order to assess odds of decision rule set, three neoteric ways were recommended to acquire the decision property. In the context of decision making, Li [15] mined latent knowledge according to suggested knowledge reduction method. The article [16] studied lower approximation reduction based on dominance relation. However, as we all know, it is inevitable that information is incomplete result from the loss and omission of information, data noise and failure of the transmission medium, and so on. Therefore, a lot of learned men use the incomplete information system as their investigative setting. On the one hand, Yang [17] adopted complement way to transform missing values into interval numbers. And then, several relative reductions were shown to obtain optimal decision rules. On the other hand, other specialists put forward new relations that are different from the equivalence relation. Some early articles [18–21] primarily introduced the concept of tolerance relation or similarity relation and its importance properties, and further study the reduction in RST. The article [22] explored problems about rule extraction with valued tolerance relation decision. In Liang and Xu [23], Liang explained the reason that the composition of attributes causes NP-hard problem and advanced a heuristic algorithm. The paper [24] defined several intuitionistic fuzzy preference relations, gave decision methods and corresponding relationships to solve the realistic problem. In a word, incomplete interval-valued decision information system is the generalization of classical information system.

Calculating approximation sets is a pivotal point for further computing all kinds of reductions or rules acquisition, which are vital investigative issues in the RST. However, the information system we studied usually be covered by new data set since the collected data set may varies with time every day. Incremental learning is a significant research direction in machine learning field. It can study new knowledge of new data set and retain knowledge learned from old data set without accessing old data set. Incremental algorithms can largely lessen the burden of calculation, while some variation appears. In other words, we utilize the original approximations and incremental algorithms to acquire updated approximation sets for partially modify the original equivalence classes(or tolerance classes) when attribute set remains unchanged in information system, which is valid for greatly improving the computational efficiency. In here, we consider three influencing factors: the attribute values, the object set, and the attribute set. They mainly introduce four situations:

- The attribute values of IS change with time, and other factors remain unchanged;
- The object set of IS varies with time, and other factors remain unchanged;
- The attribute set of IS changes, and other factors remain unchanged;
- The object set and attribute set of IS change, and other factors remain unchanged.

Researches on these issues have been studied by many experts in terms of incremental algorithms with dynamic data in recent years. The first case, the writing [25] constructed a feature matrix that arises from changes after modifying feature values and put forward incremental algorithms to calculate the second and sixth approximate sets of the set. After the attribute value in the information system changes with time, the essay [26] raised a method to update the approximate set. The second case, Yu [27,28] studied updated approximation sets under different researching contexts and the variation of objects. Based on the analytical two set-valued information systems, Luo [29] developed a mechanism to update approximations. Moreover, when many objects vary with time, the literature [30] discussed the means of updating approximation sets in dominance-based rough sets approach(DRSA). The third case, similar to the principle of change in the dominance relation, Li [31] studied methods and algorithms to compute approximation sets in DRSA. Jing [32] took advantage of knowledge granularity for replacing attribute reductions. When new information arrives, the attribute set may be generalized, the characteristic relation was studied and the updating mechanism of the approximate set was posed by Li [33]. As for the fourth case, in Hu et al. [34], Hu discussed algorithms about the variety of approximations, while the double universe changes. Combination of rough set and fuzzy set provided an effective way for replacing approximations in essay [35]. In summary, these studies help researchers dynamically updating approximations in different information systems and reducing its computation time.

The purpose of this article is to propose incremental algorithms for computing approximations when objects vary with time, but attribute set remains unchanged in IIDIS, which greatly reduces the time to compute approximation sets. In order to understand this article better, this part mainly describes some basic knowledge about the rough set and several definitions about intervals in Sect. 2. The section defines a new tolerance relation for classifying objects and establishes a rough set model of IIDIS in Sect. 3. Approaches of statically/dynamically updating approximations are proposed in Sect. 4. Then, three algorithms are designed in Sect. 5, namely the static algorithm, the dynamic algorithm when objects increase and the dynamic algorithm when objects decrease. It exhibits results of the static algorithm and the two dynamic algorithms about calculating the approximation sets concerning six data sets from UCI in Sect. 6, respectively. The conclusion of this paper is as shown in Sect. 7.

## 2 Preliminaries about rough set theory

This section first gives a few fundamental notions about RST in IIDIS for the sake of the discussion later [9,17].

In many ways, information and knowledge can be reflected in a table that uses object set as rows and attribute set as columns, which is usually known as an information system. In general, an information system is denoted as a quadruple $IS = (U, A, V, f)$. When $A = C \cup D$ and $C \cap D \neq \emptyset$ hold simultaneously, $DT = (U, C \cup D, V, f)$ is known as a decision table. It also can be called a decision information system. Here, $U$ is called universe/discourse that includes studied objects. Set of characteristics represented by discourse is usually called attribute set $A$, which contains two parts: the set of condition attributes $C$ and the set of decision attributes $D$. $V_a$ is a subset of $V$ that is the domain of attributes. It can be written as $V = \prod_{a \in A} V_a$. $f : U \times A \rightarrow V$ is a mapping to transform an ordered pair $(x, a)$ to a value for each $x \in U, a \in A$. And the mapping is called an information function. Specially, $f(x, d)(d \in D)$ is single-valued for every $x \in U$.

In mathematics, any subset R of the product $U \times U$ of the universe $U$ can be known as a binary relation on $U$. $R$ is usually referred to as an equivalence relation or indiscernibility relation on $U$ if and only if R satisfies reflexivity, symmetry, and transitivity. Pawlak approximation space can be denoted by a binary group $(U, R)$. Another mathematical object is the partition on the universe $U$, which is closely related to the equivalence relation. Specifically, a quotient set is the set of all equivalence classes obtained from the equivalence relation $R$. It can be easily verified that the quotient set is a partition on $U$, written down as $U/R = \{[x]_R | x \in U\}$, where the equivalence class $[x]_R = \{y \in U | xRy\}$ for $x \in U$.

If for $\forall a \in A, x \in U$, attribute value($f(x, a) = [l^-, l^+]$) is an interval of reals between $l^-$ and $l^+$, then $IS$ is an interval-valued information system, referred to as $IIS = (U, A, V, f)$. Particularly, if $l^- = l^+$, $f(x, a)$ is a real number, so the interval-valued information system is the generalization of classical information system, where $l^-, l^+ \in R$, $R$ is the set of real number.

Let $f(x, a) = [l^-, l^+]$, if at least one of lower bound $l^-$ and upper bound $l^+$ is an unknown value, thus, we will write down as $f(x, a) = *$. And $IIIS = (U, A, V, f)$ is an incomplete interval-valued information system. *IIDIS* $= (U, C \cup D, V, f)$ is an incomplete interval-valued decision information system or incomplete interval-valued decision table. In the following discussion, we only study the situation, where $D = \{d\}$.

The Jaccard coefficient [36] is a probability about comparing similarity and dispersion in a sample set $E$, which equals the quotient of the intersection of sample set with the union of sample set which is denoted by $J(M, N)$. Assume that $(M, N \subseteq E)$, then

$$J(E, F) = \frac{|M \cap N|}{|M \cup N|}.$$

The similarity degree of both intervals can be defined based on the above basic concepts in $IIS$.

**Definition 2.1** Given an interval-valued information system $IIS = (U, A, V, f)$, for $\forall\, a_k \in A$, $x_i, x_j \in U$. Let $f(x_i, a_k) = \mu = [\mu^-, \mu^+]$, $f(x_j, a_k) = \nu = [\nu^-, \nu^+]$, Then, the similarity degree with respect to $x_i, x_j$ under the attribute $a_k$ can be referred as

$$S_{ij}^k(\mu, \nu) = \frac{|\mu \cap \nu|}{|\mu \cup \nu|}.$$

In the above equation, $| \cdot |$ represents the length of the closed interval.

***Remark 2.1*** (1) Both the interval length of the empty set and the single-point set are equal to zero;

(2) Assume that both attribute values are single-point sets. If $\mu = \nu$, then $S_{ij}^k(\mu, \nu) = 1$; otherwise, $S_{ij}^k(\mu, \nu) = 0$;

(3) If $\mu = * \vee \nu = *$, then set the similarity degree with respect to $x_i, x_j$ equals ▲ (▲ is just a mark symbol).

**Definition 2.2** [37] Let two sets $Q$, $G$ constitute a set pair $H = (Q, G)$. According to the need of the problem $W$, we can analyze the characteristics of set pair $H$ and obtain $N$ characteristics(attributes). For two sets $Q, G$, which have same values on S attributes, different values on P attributes, and the rest of $F = N - S - P$ attribute values are ambiguous. $\frac{S}{N}$ is called the identical degree of these two sets under problem $W$, referred to as the identical degree. $\frac{P}{N}$ is called the opposite degree of these two sets under problem $W$, referred to as the opposite degree. $\frac{F}{N}$ is called the difference degree of these two sets under problem $W$, referred to as the difference degree. Then, the connection degree with respect to two sets $Q$, $G$ can be defined as

$$\mu(Q, G) = \frac{S}{N} + \frac{F}{N} \cdot i + \frac{P}{N} \cdot j.$$

which is denoted as $\mu(Q, G) = s + f \cdot i + p \cdot j$, where $s, f, p \in [0, 1]$, $s + f + p = 1$. In the calculation, set $j = -1, i \in [-1, 1]$, $i$ and $j$ also participate in the operation as coefficients. However, the functions of $i, j$ are just markings in this paper. $i$ is the marking of the difference degree, and $j$ is the marking of the opposite degree.

Given an IIDIS, the similarity degree on two objects can be calculated in the light of Definition 2.1. There are three possible situations:

(1) The two attribute values are both not equal to *, and their similarity degree is not less than a given threshold;
(2) The two attribute values are both not equal to *, and their similarity degree is less than a given threshold;
(3) At least one of the two attribute values is equal to *, and their similarity degree is considered to be ▲.

**Definition 2.3** [38] Given an incomplete interval-valued information system $IIIS = (U, A, V, f)$, $B \subseteq A$, $\forall x_i, x_j \in U$. Let $S_1 = \{b_k \in B | (S_{ij}^k(\mu, \nu) \geq \lambda) \wedge \nu \neq * \wedge \mu \neq *\}$ be a set of the attributes that the similarity degree of $x_i, x_j$ under the attribute $b_k$ is not less than a similar level $\lambda$. $P_1 = \{b_k \in B | S_{ij}^k(\mu, \nu) < \lambda \wedge \nu \neq * \wedge \mu \neq *\}$ is a set of the attributes that the similarity degree of $x_i, x_j$ under the attribute $b_k$ is less than a similar level $\lambda$. $F_1 = \{b_k \in B | S_{ij}^k(\mu, \nu) = \blacktriangle\}$ is a set of the attributes that the similarity degree of $x_i, x_j$ under the attribute $b_k$ is equal to ▲.

Furthermore, $\frac{|S_1|}{|B|}$ denotes the tolerance degree of the two objects with regard to $B$. $\frac{|P_1|}{|B|}$ denotes the opposite degree of the two objects with regard to $B$. $\frac{|F_1|}{|B|}$ denotes the difference degree of the two objects with regard to $B$. Then, the relationship of $x_i, x_j$ is known as

$$\mu_1(x_i, x_j) = \frac{|S_1|}{|B|} + \frac{|F_1|}{|B|} \cdot i + \frac{|P_1|}{|B|} \cdot j.$$

$\mu_1$ indicates similar connection degree of the two objects $x_i, x_j$. Referred to as $\mu_1(x_i, x_j) = s_1 + f_1 \cdot i + p_1 \cdot j$. Where $s_1, f_1, p_1 \in [0, 1]$, $s_1 + f_1 + p_1 = 1$, the functions of $i$, $j$ are just markings. $i$ is the marking of the difference degree, and $j$ is the marking of the opposite degree.

## 3 Rough set in IIDIS

In this section, a novel tolerance relation is defined in light of similar connection degree and constructed rough set model.

**Definition 3.1** In the incomplete interval-valued decision information system $IIDIS = (U, C \cup \{d\}, V, f)$, $\forall B \subseteq C$, for each $x_i, x_j \in U$. $\alpha \in (0.5, 1]$, $\beta, \gamma \in [0, 0.5)$. The multi-threshold tolerance relation can be referred as

$$\begin{aligned} R_B^{\alpha\beta\gamma} &= \{(x_i, x_i)\} \cup \{(x_i, x_j) | \mu_1(x_i, x_j) = s_1 + f_1 \cdot i + p_1 \cdot j, \\ &\quad s_1 \geq \alpha, f_1 \leq \gamma, p_1 \leq \beta, s_1 + f_1 + p_1 = 1\}. \end{aligned}$$

$s_1, f_1, p_1$ represent, respectively, the tolerance degree, the difference degree, and the opposite degree of objects $x_i, x_j$ with reference to attribute set $B$. $\alpha$ is the threshold of the tolerance degree, $\beta$ is the threshold of the opposite degree, and $\gamma$ is the threshold of the difference degree.

The multi-threshold tolerance class can be defined as

$$\begin{aligned} [x_i]_{R_B^{\alpha\beta\gamma}} &= \{x_j \in U | (x_i, x_j) \in R_B^{\alpha\beta\gamma}\}. \\ U/R_B^{\alpha\beta\gamma} &= \{[x_1]_{R_B^{\alpha\beta\gamma}}, [x_2]_{R_B^{\alpha\beta\gamma}}, \ldots, [x_{|U|}]_{R_B^{\alpha\beta\gamma}}\}. \end{aligned}$$

In addition, a binary relation under decision attribute $d$ is remembered as $R_d = \{(x_i, x_j) \in U^2 | f_d(x_i) = f_d(x_j)\}$. Decision class and quotient set can be alluded to as $[x]_d = \{y \in U | f_d(x) = f_d(y)\}$, $U/d = \{[x]_d | \forall x \in U\} = \{D_1, D_2, \ldots, D_q\} (D_i \subseteq U, i = 1, 2, \ldots, q)$, respectively. It can easily be proven that relation $R_d$ is an equivalence relation and $U/d$ constitutes a partition on $U$.

***Remark 3.1*** (1) It obviously observes that the multi-threshold tolerance relation is reflexive and symmetrical rather than transitive, which is a tolerance relation; $J = \cup\{[x_i]_{R_B^{\alpha\beta\gamma}}\}$ is a cover on $U$.

(2) It is reasonable to put two objects in the same class if the tolerance degree of the two objects under the attribute subset is not less than $\alpha$ and the opposite degree, the difference degree of the two objects under the attribute subset is less than or equal to $\beta$, $\gamma$, respectively.

(3) If we do not consider parameter $\gamma$ and the range of $\alpha, \beta$, the multi-threshold tolerance relation is degraded into the tolerance relation in Zeng et al. [38]. Therefore, the tolerance relation in Zeng et al. [38] can be regarded as a specific situation of multi-threshold tolerance relation.

(4) When $B = \{a\}$, $\{a\}$ can be replaced by $a$. The following paper will denote $R_a^{\alpha\beta\gamma}$, $[x_i]_{R_a^{\alpha\beta\gamma}}$, $U/R_a^{\alpha\beta\gamma}$.

Every uncertain concept can be represented by a couple of accurate notions that are upper approximation and lower approximation(for short approximations). The following gives definitions about upper/lower approximation.

**Definition 3.2** In the incomplete interval-valued decision information system $IIDIS = (U, C \cup \{d\}, V, f)$, for each $B \subseteq C$, $X \subseteq U$. The approximations of $X$ concerning a multi-threshold tolerance relation $R_B^{\alpha\beta\gamma}$ can be represented by

$$\begin{aligned} \underline{R_B^{\alpha\beta\gamma}}(X) &= \{x \in U | [x]_{R_B^{\alpha\beta\gamma}} \subseteq X\}; \\ \overline{R_B^{\alpha\beta\gamma}}(X) &= \{x \in U | [x]_{R_B^{\alpha\beta\gamma}} \cap X \neq \emptyset\}. \end{aligned}$$

$\underline{R_B^{\alpha\beta\gamma}}$, $\overline{R_B^{\alpha\beta\gamma}}$ are called lower and upper approximation operators of $X$ concerning a multi-threshold tolerance relation $R_B^{\alpha\beta\gamma}$.

Moreover, similar to classical rough set, positive region is recorded as $Pos_{R_B^{\alpha\beta\gamma}}(X) = \underline{R_B^{\alpha\beta\gamma}}(X)$; negative region is known as $Neg_{R_B^{\alpha\beta\gamma}}(X) = U - \overline{R_B^{\alpha\beta\gamma}}(X)$, what boundary region represents the difference between the lower approximation and upper approximation of $X$ concerning $R_B^{\alpha\beta\gamma}$ is denoted by $Bn_{R_B^{\alpha\beta\gamma}}(X) = \overline{R_B^{\alpha\beta\gamma}}(X) - \underline{R_B^{\alpha\beta\gamma}}(X)$.

***Example 1*** Just as revealed in Table 1 (processing method is similar with Sect. 6), it is an incomplete interval-valued decision information system. It represents situation of treating wart of 20 people. Here, $IIDIS = (U, C \cup \{d\}, V, f)$. Where universe $U = \{x_1, x_2, \ldots, x_{20}\}$, $x_i$ represent the $i$th people ($i = 1, 2, \ldots, 20$). $A = \{a_1, a_2, \ldots, a_7\}$, $a_i (i = 1, 2, \ldots, 7)$ represent sex, age, time, number of warts, type, area, induration diameter, respectively. $d$ shows the result of treatment. $f(x, d) \in \{0, 1\}$. In this example, let $\lambda = 0.5, \alpha = 0.6, \beta = 0.4, \gamma = 0.2$.

It is easy to know decision attribute divides discourse into two parts, $U/d = \{D_1, D_2\}$, where $D_1 \cup D_2 = U$, $D_1 \cap D_2 = \emptyset$. Assume that $D_1 = \{x_5, x_{10}, x_{13}, x_{14}, x_{15}\}$, then $D_2 = U - D_1$. Let $B_1 \subseteq C$, $B_1 = \{a_7\}$. It can be obtained by calculation:

$$[x_1]_{R_{B_1}^{\alpha\beta\gamma}} = [x_4]_{R_{B_1}^{\alpha\beta\gamma}} = [x_{13}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_1, x_4, x_{13}\};$$
$$[x_2]_{R_{B_1}^{\alpha\beta\gamma}} = [x_5]_{R_{B_1}^{\alpha\beta\gamma}} = [x_9]_{R_{B_1}^{\alpha\beta\gamma}} = [x_{10}]_{R_{B_1}^{\alpha\beta\gamma}} = [x_{11}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_2, x_5, x_9, x_{10}, x_{11}\};$$
$$[x_3]_{R_{B_1}^{\alpha\beta\gamma}} = [x_{19}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_3, x_{19}\}; [x_6]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_6\}; [x_7]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_7\};$$
$$[x_8]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_8\}; [x_{12}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_{12}\}; [x_{14}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_{14}\}; [x_{15}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_{15}\};$$
$$[x_{16}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_{16}\}; [x_{17}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_{17}\}; [x_{18}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_{18}\}; [x_{20}]_{R_{B_1}^{\alpha\beta\gamma}} = \{x_{20}\};$$

Hence, according to definitions of approximations:

$$\underline{R_{B_1}^{\alpha\beta\gamma}}(D_1) = \{x_{14}, x_{15}\},$$
$$\overline{R_{B_1}^{\alpha\beta\gamma}}(D_1) = \{x_1, x_2, x_4, x_5, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}\},$$

So the positive region is $Pos_{R_{B_1}^{\alpha\beta\gamma}}(D_1) = \{x_{14}, x_{15}\}$,

negative region is $Neg_{R_{B_1}^{\alpha\beta\gamma}}(D_1) = \{x_3, x_6, x_7, x_8, x_{12}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}$,

and boundary region is $Bn_{R_{B_1}^{\alpha\beta\gamma}}(D_1) = \{x_1, x_2, x_4, x_5, x_9, x_{10}, x_{11}, x_{13}\}$.

## 4 Method for obtaining approximations with dynamic data in IIDIS

As society modernizes and progresses, people have entered the era of information. Data analysis and data mining are hot topics in the research of rough set. We can know some knowledge according to the raw data. However, the object set discussed by these data may increases or decreases for some reasons over time. Incremental machine learning is a vital method to deal with dynamic data in data mining. This section will focus on how to solve variation of approximations when some objects add to or delete from the initial data, but attribute set remains unchanged in IIDIS. In the following discussion, we only consider an object that adds to or deletes from the initial data and obtain new data in order to facilitate the calculation. As for increasing or decreasing multiple objects, we only need to repeat the updating principle of individual object step by step. In short, we need to observe two incomplete interval-valued decision information systems as our researching objects. One is the initial system $IIDIS = \{U, C \cup \{d\}, V, f\}$, and the other is the testing system $IIDIS' = \{U', C \cup \{d\}, V', f\}$.

**Table 1** Incomplete interval-valued decision information system

| $U$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $d$ |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | [1.90, 2.10] | [31.35, 34.65] | * | * | * | * | [23.75, 26.25] | 1 |
| $x_2$ | [0.95, 1.05] | [23.75, 26.25] | [5.4625, 6.0375] | [1.90, 2.10] | [0.95, 1.05] | [285, 315] | [6.65, 7.35] | 1 |
| $x_3$ | [1.90, 2.10] | * | * | * | [0.95, 1.05] | [28.50, 31.50] | [2.85, 3.15] | 1 |
| $x_4$ | [1.90, 2.10] | [45.60, 50.40] | [9.7375, 10.7625] | [6.65, 7.35] | * | [47.50, 52.50] | [23.75, 26.25] | 1 |
| $x_5$ | [0.95, 1.05] | [31.35, 34.65] | [1.6625, 1.8375] | [6.65, 7.35] | [1.90, 2.10] | [360.05, 397.95] | [6.65, 7.35] | 0 |
| $x_6$ | [1.90, 2.10] | [36.10, 39.90] | [2.3750, 2.6250] | [0.95, 1.05] | * | [40.85, 45.15] | [47.50, 52.50] | 1 |
| $x_7$ | [0.95, 1.05] | * | [9.500, 10.5000] | * | * | * | * | 1 |
| $x_8$ | * | [22.80, 25.20] | [4.0375, 4.4625] | [0.95, 1.05] | [0.95, 1.05] | * | [28.50, 31.50] | 1 |
| $x_9$ | [0.95, 1.05] | [18.05, 19.95] | [7.3625, 8.1375] | * | [0.95, 1.05] | * | [6.65, 7.35] | 1 |
| $x_{10}$ | [0.95, 1.05] | [32.30, 35.70] | * | [6.65, 7.35] | * | [60.80, 67.20] | [6.65, 7.35] | 0 |
| $x_{11}$ | [0.95, 1.05] | [27.55, 30.45] | [4.7500, 5.2500] | [11.40, 12.60] | [2.85, 3.15] | [71.25, 78.75] | [6.65, 7.35] | 1 |
| $x_{12}$ | [0.95, 1.05] | * | [2.1375, 2.3625] | * | [2.85, 3.15] | [48.45, 53.55] | * | 1 |
| $x_{13}$ | [0.95, 1.05] | [43.70, 48.30] | * | [3.80, 4.20] | * | [86.45, 95.55] | [23.75, 26.25] | 0 |
| $x_{14}$ | * | * | * | * | * | [82.65, 91.35] | [5.70, 6.30] | 0 |
| $x_{15}$ | [0.95, 1.05] | * | [10.6875, 11.8125] | * | [0.95, 1.05] | [68.40, 75.60] | * | 0 |
| $x_{16}$ | * | [16.15, 17.85] | [8.0750, 8.9250] | [1.90, 2.10] | * | * | [7.60, 8.40] | 1 |
| $x_{17}$ | * | * | [4.7500, 5.2500] | [1.90, 2.10] | [0.95, 1.05] | * | [4.75, 5.25] | 1 |
| $x_{18}$ | [1.90, 2.10] | [21.85, 24.15] | [6.4125, 7.0875] | [5.70, 6.30] | [0.95, 1.05]] | * | [1.90, 2.10] | 1 |
| $x_{19}$ | * | [34.20, 37.80] | [1.6625, 1.8375] | * | [2.85, 3.15] | [42.75, 47.25] | [2.85, 3.15] | 1 |
| $x_{20}$ | [0.95, 1.05] | [36.10, 39.90] | [7.1250, 7.8750] | [7.60, 8.40] | [1.90, 2.10] | [53.20, 58.80] | [42.75, 47.25] | 1 |

### 4.1 Method for obtaining approximations with deleting an object in IIDIS

Now, we can acquire an initial system $IIDIS = \{U, C \cup \{d\}, V, f\}$ by disposing raw data. Testing system $IIDIS' = \{U', C \cup \{d\}, V', f\}$ can be obtained by deleting an object from the discourse $U$, and both the multi-threshold tolerance classes $[x]_{R_B^{\alpha\beta\gamma}}$ (for any $x \in U$, $B \subseteq C$) and decision classes $D_j (j = 1, 2, \ldots, q)$ will change at the same time, where $U' = U - \{x_0\}$. We explore the method for solving approximation problem and discuss whether the deleted object $x_0$ belongs to $D_j$.

1. The first case: if the deleted object $x_0 \in D_j$, then $D'_j = D_j - \{x_0\}$.

**Proposition 4.1** *Given an incomplete interval-valued decision information system* $IIDIS = (U, C \cup \{d\}, V, f)$, *for each* $B \subseteq C$. *If* $x_0 \in D_j$ $(j \in \{1, 2, \ldots, q\})$, *then the variation of approximations of* $D_j$ *can be updated as:*

(1) *If* $x_0 \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$, *then* $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) - \{x_0\}$. *If not,* $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j)$.

(2) $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = (\overline{R_B^{\alpha\beta\gamma}}(D_j) - [x_0]_{R_B^{\alpha\beta\gamma}}) \cup \Delta$, *where* $\Delta = \{x | [x]_{R_B^{\alpha\beta\gamma}} \cap D'_j \neq \emptyset, x \in ([x_0]_{R_B^{\alpha\beta\gamma}} - \{x_0\})\}$.

***Proof*** (1) If $x_0 \in D_j$, there have $U' = U - \{x_0\}$, $D'_j = D_j - \{x_0\}$. For $\forall x \in U'$, $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} - \{x_0\}$. On account of $x_0 \in D_j$, when $[x]_{R_B^{\alpha\beta\gamma}} \subseteq D_j$, $[x]'_{R_B^{\alpha\beta\gamma}} \subseteq D'_j$ holds. Likewise, when $[x]_{R_B^{\alpha\beta\gamma}} \nsubseteq D_j$, $[x]'_{R_B^{\alpha\beta\gamma}} \nsubseteq D'_j$ holds. In the light of Definition 3.2, apparently, for each $x \in U'$, if $x \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$, then $x \in \underline{R_B^{\alpha\beta\gamma}}(D'_j)$, and if $x \notin \underline{R_B^{\alpha\beta\gamma}}(D_j)$, then $x \notin \underline{R_B^{\alpha\beta\gamma}}(D_j)'$. Therefore, if $x_0 \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$, then $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) - \{x_0\}$. Otherwise, $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j)$.

(2) In the light of Definition 3.2, the upper approximation of $D_j$ is $\overline{R_B^{\alpha\beta\gamma}}(D_j) = \{x \in U | [x]_{R_B^{\alpha\beta\gamma}} \cap D_j \neq \emptyset\}$. The set $[x_0]_{R_B^{\alpha\beta\gamma}}$ should be deleted from $\overline{R_B^{\alpha\beta\gamma}}(D_j)$, while $[x_0]_{R_B^{\alpha\beta\gamma}} \cap D_j = [x_0]_{R_B^{\alpha\beta\gamma}}$. But if there exits an object $x \in [x_0]_{R_B^{\alpha\beta\gamma}} - \{x_0\}$, s.t. $[x]_{R_B^{\alpha\beta\gamma}} \cap D'_j \neq \emptyset$. These objects like $x$ should not be deleted. So $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) \cup \Delta$. Where $\Delta = \{x | [x]_{R_B^{\alpha\beta\gamma}} \cap D'_j \neq \emptyset, x \in ([x_0]_{R_B^{\alpha\beta\gamma}} - \{x_0\})\}$. □

2. The second case: if the deleted object $x_0 \notin D_j$, then $D'_j = D_j$.

**Proposition 4.2** *Given an incomplete interval-valued decision information system* $IIDIS = (U, C \cup \{d\}, V, f)$, *for each* $B \subseteq C$. *If* $x_0 \notin D_j$ $(j \in \{1, 2, \ldots, q\})$, *then the variation of approximations of* $D_j$ *can be updated as:*

(1) $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) \cup \Delta$, *where* $\Delta = \{x | ([x]_{R_B^{\alpha\beta\gamma}} - \{x_0\}) \subseteq D_j, x \in (D_j - \underline{R_B^{\alpha\beta\gamma}}(D_j))\}$.

(2) *If* $x_0 \in \overline{R_B^{\alpha\beta\gamma}}(D_j)$, *then* $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) - \{x_0\}$. *If not,* $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j)$.

***Proof*** (1) If $x_0 \notin D_j$, there have $U' = U - \{x_0\}$, $D'_j = D_j$. For $\forall x \in U'$, $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} - \{x_0\}$. Supposing that $[x]_{R_B^{\alpha\beta\gamma}} \subseteq D_j$, hence $[x]'_{R_B^{\alpha\beta\gamma}} \subseteq D'_j$. According to Definition 3.2, for

$\forall x \in D_j$, if $x \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$, then $[x]'_{R_B^{\alpha\beta\gamma}} \subseteq [x]_{R_B^{\alpha\beta\gamma}} \subseteq D_j = D'_j$, namely $[x]'_{R_B^{\alpha\beta\gamma}} \subseteq D'_j$, so we can gain $\underline{R_B^{\alpha\beta\gamma}}(D_j) \subseteq \underline{R_B^{\alpha\beta\gamma}}(D'_j)$.

Assume that $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) \cup \Delta$. For $\forall x \in D_j - \underline{R_B^{\alpha\beta\gamma}}(D_j)$, there have $[x]_{R_B^{\alpha\beta\gamma}} \nsubseteq D_j$ but may exist $x_0 \in [x]_{R_B^{\alpha\beta\gamma}}$, s.t. $([x]_{R_B^{\alpha\beta\gamma}} - \{x_0\}) \subseteq D_j$. These objects like $x$ cannot be deleted. Therefore, $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) \cup \Delta$, where $\Delta = \{x | ([x]_{R_B^{\alpha\beta\gamma}} - \{x_0\}) \subseteq D_j, x \in (D_j - \underline{R_B^{\alpha\beta\gamma}}(D_j))\}$.

(2) On the one hand, if $x_0 \in \overline{R_B^{\alpha\beta\gamma}}(D_j)$, while $x_0 \notin D_j$, according to Definition 3.2, there exists at least an object $x \in D_j$, s.t. $x_0 \in [x]_{R_B^{\alpha\beta\gamma}}$. After deleting $x_0$, $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} - \{x_0\}$ and $[x]'_{R_B^{\alpha\beta\gamma}} \cap D_j \neq \emptyset$ hold. Thus, $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) - \{x_0\}$. On the other hand, if $x_0 \notin \overline{R_B^{\alpha\beta\gamma}}(D_j)$, then for $\forall x \in D_j$, $x_0 \notin [x]_{R_B^{\alpha\beta\gamma}}$. So $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j)$. □

### 4.2 Method for obtaining approximations with adding an object in IIDIS

We can obtain initial system $IIDIS = \{U, C \cup \{d\}, V, f\}$ after dealing with raw data. Testing system $IIDIS' = \{U', C \cup \{d\}, V', f\}$ can be acquired by adding an object to the universe $U$, and both the multi-threshold tolerance classes $[x]_{R_B^{\alpha\beta\gamma}}$ (for any $x \in U$, $B \subseteq C$) and decision classes $D_j (j = 1, 2, \ldots, q)$ evolve over time, where $U' = U \cup \{x_0\}$ and the added object is $x_0$. There appear three cases with regard to the upper and lower approximation of $D_j$:

- An object $x_0$ is added to the universe $U$, and there exists $x \in D_j$, s.t. $f(x, d) = f(x_0, d)$ holds, thus $D'_j = D_j \cup \{x_0\}$.
- An object $x_0$ is added to the universe $U$. There exists $x \in (U - D_j)$, s.t. $f(x, d) = f(x_0, d)$ holds. Besides, for each $x \in D_j$, $f(x, d) \neq f(x_0, d)$ holds; thus, $D'_j = D_j$.
- An object $x_0$ is added to the universe $U$ and for $\forall x \in U$, $f(x, d) \neq f(x_0, d)$ holds, and thus, there engenders a novel decision class $D_{q+1} = \{x_0\}$.

1. The first case: $D'_j = D_j \cup \{x_0\}$.

**Proposition 4.3** *Given an incomplete interval-valued decision information system $IIDIS = (U, C \cup \{d\}, V, f)$, for each $B \subseteq C$. If an object $x_0$ is added to the universe $U$, there exists $x \in D_j (j = 1, 2, \ldots, q)$, such that $f(x, d) = f(x_0, d)$ holds, and thus, the variation of approximations of $D_j$ can be updated as:*

(1) *If $[x_0]_{R_B^{\alpha\beta\gamma}} \subseteq D'_j$, then $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) \cup \{x_0\}$. If not, $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j)$.*

(2) $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) \cup [x_0]_{R_B^{\alpha\beta\gamma}}$.

***Proof*** (1) When an object $x_0$ adds to the universe $U$, the universe will be turned into $U' = U \cup \{x_0\}$. For $\forall x \in D_j$, if $[x]_{R_B^{\alpha\beta\gamma}} \subseteq D_j$, then $x \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$. Hence, if for $x \in D_j$, $x_0 \in [x]_{R_B^{\alpha\beta\gamma}}$ holds, then $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} \cup \{x_0\}$. Namely, if $[x]_{R_B^{\alpha\beta\gamma}} \subseteq D_j$, it can obtain $[x]'_{R_B^{\alpha\beta\gamma}} \subseteq D'_j$. Similarly, if $[x]_{R_B^{\alpha\beta\gamma}} \nsubseteq D_j$, it can obtain $[x]'_{R_B^{\alpha\beta\gamma}} \nsubseteq D'_j$. Therefore, if $x \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$, then $x \in \underline{R_B^{\alpha\beta\gamma}}(D'_j)$ holds. If $x \notin \underline{R_B^{\alpha\beta\gamma}}(D_j)$, then $x \notin \underline{R_B^{\alpha\beta\gamma}}(D'_j)$ holds. We can assume that $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) \cup \Delta$. If $[x_0]_{R_B^{\alpha\beta\gamma}} \subseteq D'_j$, then $x_0 \in \underline{R_B^{\alpha\beta\gamma}}(D'_j)$. So $\Delta = \{x_0\}$, $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) \cup \{x_0\}$. Otherwise, $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j)$.

(2) In the light of Definition 3.2, the upper approximation of $D_j$ is $\overline{R_B^{\alpha\beta\gamma}}(D_j) = \{x \in U | [x]_{R_B^{\alpha\beta\gamma}} \cap D_j \neq \emptyset\}$. For each $x \in U$, $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} \cup \{x_0\}$ or $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}}$ hold. Suppose that $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) \cup \Delta$, because $[x_0]_{R_B^{\alpha\beta\gamma}} \cap D'_j \neq \emptyset$ must be established and $R_B^{\alpha\beta\gamma}$ satisfies the quality of symmetry. Then, $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) \cup [x_0]_{R_B^{\alpha\beta\gamma}}$. □

2. The second case: $D'_j = D_j$.

**Proposition 4.4** *Given an incomplete interval-valued decision information system IIDIS $= (U, C \cup \{d\}, V, f)$, for each $B \subseteq C$. There exists $x \in (U - D_j)$, s.t. $f(x, d) = f(x_0, d)$ holds and for each $x \in D_j$, $f(x, d) \neq f(x_0, d)$ also holds, and thus, the variation of approximations of $D_j$ can be updated as:*

(1) $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) - \Delta$, *where* $\Delta = \{x | x \in \underline{R_B^{\alpha\beta\gamma}}(D_j), x_0 \in [x]'_{R_B^{\alpha\beta\gamma}}\}$.

(2) *If there exists $x \in D_j$, s.t. $x_0 \in [x]'_{R_B^{\alpha\beta\gamma}}$ holds, thus $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) \cup \{x_0\}$. If not, $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j)$.*

***Proof*** (1) When an object $x_0$ adds to the universe $U$, the universe will be turned into $U' = U \cup \{x_0\}$. For each $x \in D'_j$, there have $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}}$ or $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} \cup \{x_0\}$. If $[x]_{R_B^{\alpha\beta\gamma}} \nsubseteq D_j$, then $[x]'_{R_B^{\alpha\beta\gamma}} \nsubseteq D'_j$. That is, $x \in \underline{R_B^{\alpha\beta\gamma}}(D'_j)$, then $x \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$. Hence, we can assume that $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) - \Delta$ and only consider $x \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$. When an object $x_0$ adds in the universe $U$, there may be $\exists x \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$ and $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} \cup \{x_0\}$, thus $[x]'_{R_B^{\alpha\beta\gamma}} \nsubseteq D_j$, namely $x \notin \underline{R_B^{\alpha\beta\gamma}}(D'_j)$. Therefore, $\underline{R_B^{\alpha\beta\gamma}}(D_j)' = \underline{R_B^{\alpha\beta\gamma}}(D_j) - \Delta$, where $\Delta = \{x | x \in \underline{R_B^{\alpha\beta\gamma}}(D_j), x_0 \in [x]'_{R_B^{\alpha\beta\gamma}}\}$.

(2) If $\exists x \in D_j, s.t. x_0 \in [x]'_{R_B^{\alpha\beta\gamma}}$(that is, $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} \cup \{x_0\}$), then $x_0 \in \overline{R_B^{\alpha\beta\gamma}}(D'_j)$, namely $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) \cup \{x_0\}$. Otherwise, if for $\forall x \in D_j$, $x_0 \notin [x]'_{R_B^{\alpha\beta\gamma}}$ holds. Then, $[x]'_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}}$. So $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j)$. □

3. The third case: An object $x_0$ is added to the universe $U$ and for each $x \in U$, $f(x, d) \neq f(x_0, d)$ holds, and thus, there engenders a novel decision class $D_{q+1} = \{x_0\}$.

**Proposition 4.5** *Given an incomplete interval-valued decision information system IIDIS $= (U, C \cup \{d\}, V, f)$, for each $B \subseteq C$. Then, the approximations of $D_{q+1}$ can be represented by:*

(1) *If $[x_0]_{R_B^{\alpha\beta\gamma}} = \{x_0\}$, then $\underline{R_B^{\alpha\beta\gamma}}(D_{q+1}) = \{x_0\}$. If not, $\underline{R_B^{\alpha\beta\gamma}}(D_{q+1}) = \emptyset$.*

(2) $\overline{R_B^{\alpha\beta\gamma}}(D_{q+1}) = [x_0]_{R_B^{\alpha\beta\gamma}}$.

***Proof*** (1) $D_{q+1}$ is a novel decision class. For each $x \in U$, $f(x, d) \neq f(x_0, d)$ holds, so $D_{q+1} = \{x_0\}$. If $[x_0]_{R_B^{\alpha\beta\gamma}} = \{x_0\}$, then $\underline{R_B^{\alpha\beta\gamma}}(D_{q+1}) = \{x_0\}$. Otherwise, $\underline{R_B^{\alpha\beta\gamma}}(D_{q+1}) = \emptyset$.

(2) The multi-threshold tolerance relation is symmetrical in the light of Remark 3.1; then, $x \in [x_0]_{R_B^{\alpha\beta\gamma}}$ be equivalent to $x_0 \in [x]_{R_B^{\alpha\beta\gamma}}$. Because $D_{q+1} = \{x_0\}$, thus, $\overline{R_B^{\alpha\beta\gamma}}(D_{q+1})$=$\{x|[x]_{R_B^{\alpha\beta\gamma}} \cap D_{q+1} \neq \emptyset\}$=$\{x|[x]_{R_B^{\alpha\beta\gamma}} \cap \{x_0\} \neq \emptyset\}$=$[x_0]_{R_B^{\alpha\beta\gamma}}$. □

## 5 A static algorithm and two dynamic algorithms for obtaining approximations in IIDIS with changed objects

If the universe changes, the approximations of an uncertainty notion also vary. One way is to find approximation sets step by step based on Definition 3.2 after changing objects. Another method is to update approximation sets according to the prior knowledge and original approximation sets when the universe varies. In this section, we firstly display the static algorithm for computing approximations according to Definition 3.2 in IIDIS. Then, dynamic algorithms for computing approximations are presented in IIDIS when the universe changes based on above-mentioned five Propositions. The following will explain these algorithms in detail.

### 5.1 The static algorithm for obtaining approximations in IIDIS

In there, we devise an algorithm according to definition of approximations for obtaining approximations after changing objects in IIDIS, which is called static algorithm(that is, Algorithm 1). Firstly, we introduce a testing system $IIDIS = (U, C \cup \{d\}, V, f)$ after changing object set. In steps 2–5, we compute decision classes $D_j(j = 1, 2, \ldots, q\}$ and multi-threshold tolerance classes $[x]_{R_B^{\alpha\beta\gamma}}$ for every $x \in U$. Steps 6–8 initialize lower and upper approximations to empty set. Next, in steps 10–17, the approximations of decision class $D_j$ directly be calculated in line with Definition 3.2. The time complexity of Algorithm 1 can be seen in Table 2, which describes the time complexity of the best case (The Best Complexity) and the worst case (The Worst Complexity).

### 5.2 The dynamic algorithm about variation of approximations in IIDIS while removing an object

We design a dynamic algorithm in the light of Proposition 4.1 and 4.2 for gaining approximations after removing an object from the universe $U$ in IIDIS, which is Algorithm 2. We introduce an initial system $IIDIS = (U, C \cup \{d\}, V, f)$, the initial multi-threshold tolerance classes $[x]_{R_B^{\alpha\beta\gamma}}$ for every $x \in U$, initial quotient set $U/d = \{D_1, D_2, \ldots, D_q\}$, and the initial approximations of $D_j$: $\underline{R_B^{\alpha\beta\gamma}}(D_j)$, $\overline{R_B^{\alpha\beta\gamma}}(D_j)(i = 1, 2, \ldots, q)$. After selecting a removal object $x_0$, the whole Algorithm 2 mainly includes two parts. On the one hand, in steps 4–14, Algorithm 2 introduces procedures with regard to updating lower and upper approximations of decision class $D_j$ when the removal object $x_0$ belongs to $D_j$, where steps 4–8 count the updated lower approximation and steps 9–14 compute the updated upper approximation by Proposition 4.1. On the other hand, in steps 16–29, Algorithm 2 describes processes concerning updated approximations of decision class $D_j$, while the removal object $x_0$ does not belong to $D_j$, where steps 16–24 calculate the updated lower approximation and steps 25–29 compute the updated upper approximation by Proposition 4.2. In order to directly understand

**Algorithm 1:** The static algorithm for obtaining approximations in IIDIS

**Input** : a testing system $IIDIS = (U, C \cup \{d\}, V, f)$, where $B \subseteq C$.
**Output** : the approximations of decision class $D_j$ in IIDIS.

1 **begin**
2   compute $U/d = \{D_1, D_2, \ldots, D_q\}$; `/* compute decision classes` $D_j$ `*/`
3   **for** $x \in U$ **do**
4     compute $[x]_{R_B^{\alpha\beta\gamma}}$;
5   **end**
6   **for** *j=1:q* **do**
7     let $\underline{R_B^{\alpha\beta\gamma}}(D_j) = \emptyset; \overline{R_B^{\alpha\beta\gamma}}(D_j) = \emptyset$; `/* the initialization of lower and upper approximations */`
8   **end**
9   **for** *j=1:q* **do**
10     **for** $x \in U$ **do**
11       **if** $[x]_{R_B^{\alpha\beta\gamma}} \subseteq D_j$ **then**
12         $\underline{R_B^{\alpha\beta\gamma}}(D_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) \cup \{x\}$;
13       **end**
14       **if** $x \in D_j$ **then**
15         $\overline{R_B^{\alpha\beta\gamma}}(D_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) \cup [x]_{R_B^{\alpha\beta\gamma}}$; `/* calculate the lower/upper approximation of` $D_j$ `by Definition 3.2 */`
16       **end**
17     **end**
18   **end**
  **return** : $\underline{R_B^{\alpha\beta\gamma}}(D_j), \overline{R_B^{\alpha\beta\gamma}}(D_j)$.
19 **end**

**Table 2** The time complexity of Algorithm 1

| Lines | The best complexity | The worst complexity |
|---|---|---|
| 2 | $\Theta(\lvert U\rvert)$ | $\Theta(\lvert U\rvert)$ |
| 3–5 | $\Theta(\lvert B\rvert\lvert U\rvert^2)$ | $\Theta(\lvert B\rvert\lvert U\rvert^2)$ |
| 6–8 | $\Theta(q)$ | $\Theta(q)$ |
| 9–18 | $\Theta(q\lvert U\rvert)$ | $\Theta(\lvert U\rvert^2)$ |
| Total | $\Theta(\lvert B\rvert\lvert U\rvert^2)$ | $\Theta(\lvert B\rvert\lvert U\rvert^2)$ |

this algorithm, the flowchart of Algorithm 2 is displayed in Fig. 1. The time complexity of Algorithm 2 is shown in Table 3 that introduces the time complexity of the best case (The Best Complexity) and the worst case (The Worst Complexity).

### 5.3 The dynamic algorithm about variation of approximations in IIDIS while adding an object

We plan a dynamic algorithm in the light of Proposition 4.3–4.5 for acquiring approximations after adding an object to the universe $U$ in IIDIS, which is Algorithm 3. We input an initial system $IIDIS = (U, C \cup \{d\}, V, f)$, the initial multi-threshold tolerance classes $[x]_{R_B^{\alpha\beta\gamma}}$ for every $x \in U$, initial quotient set $U/d = \{D_1, D_2, \ldots, D_q\}$, and the initial approximations of $D_j$: $\underline{R_B^{\alpha\beta\gamma}}(D_j), \overline{R_B^{\alpha\beta\gamma}}(D_j)(i = 1, 2, \ldots, q)$. After selecting an inserted object $x_0$, the whole

**Algorithm 2:** The dynamic algorithm to compute approximations after deleting an object in IIDIS

**Input** : (1)the initial system $IIDIS = (U, C \cup \{d\}, V, f)$, where $B \subseteq C$.
(2)the initial multi-threshold tolerance classes $[x]_{R_B^{\alpha\beta\gamma}} (\forall x \in U)$ and initial quotient set $U/d = \{D_1, D_2, \ldots, D_q\}$.
(3)the initial lower and upper approximations of $D_j$: $\underline{R_B^{\alpha\beta\gamma}}(D_j)$, $\overline{R_B^{\alpha\beta\gamma}}(D_j)(j = 1, 2, \ldots, q)$.
(4)the deleted object: $x_0$.

**Output** : the approximations of decision class $D_j$ after deleting $x_0$: $\underline{R_B^{\alpha\beta\gamma}}(D'_j)$, $\overline{R_B^{\alpha\beta\gamma}}(D'_j)$.

**begin**
  **for** *j=1:q* **do**
    **if** $x_0 \in D_j$ **then**
      **if** $x_0 \in \underline{R_B^{\alpha\beta\gamma}}(D_j)$ **then**
        $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j) - \{x_0\}$;
      **else**
        $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j)$;
      **end**
      $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) - [x_0]_{R_B^{\alpha\beta\gamma}}$;
      **for** $x \in ([x_0]_{R_B^{\alpha\beta\gamma}} - \{x_0\})$ **do**
        **if** $[x]_{R_B^{\alpha\beta\gamma}} \cap (D_j - \{x_0\}) \neq \emptyset$ **then**
          $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D'_j) \cup \{x\}$; /* compute the updated lower/upper approximation of $D_j$ by Proposition 4.1 */
        **end**
      **end**
    **else**
      let $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D_j)$;
      **for** $x \in (D_j - \underline{R_B^{\alpha\beta\gamma}}(D_j))$ **do**
        **if** $x_0 \in [x]_{R_B^{\alpha\beta\gamma}}$ **then**
          $[x]_{R_B^{\alpha\beta\gamma}} = [x]_{R_B^{\alpha\beta\gamma}} - \{x_0\}$;
        **end**
        **if** $[x]_{R_B^{\alpha\beta\gamma}} \subseteq D_j$ **then**
          $\underline{R_B^{\alpha\beta\gamma}}(D'_j) = \underline{R_B^{\alpha\beta\gamma}}(D'_j) \cup \{x\}$;
        **end**
      **end**
      **if** $x_0 \in \overline{R_B^{\alpha\beta\gamma}}(D_j)$ **then**
        $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j) - \{x_0\}$;
      **else**
        $\overline{R_B^{\alpha\beta\gamma}}(D'_j) = \overline{R_B^{\alpha\beta\gamma}}(D_j)$; /* compute the updated lower/upper approximation of $D_j$ by Proposition 4.2 */
      **end**
    **end**
  **end**
  **return** : $\underline{R_B^{\alpha\beta\gamma}}(D'_j)$,$\overline{R_B^{\alpha\beta\gamma}}(D'_j)$.
**end**

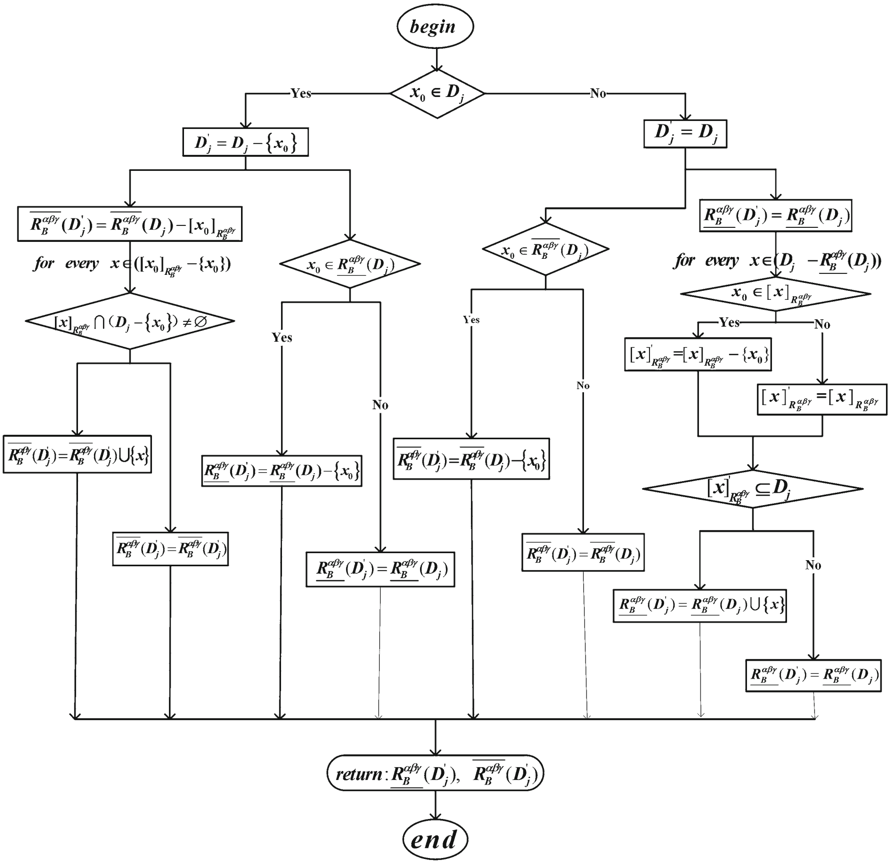


**Fig. 1** The flowchart of Algorithm 2

**Table 3** The time complexity of Algorithm 2

| Lines | The best complexity | The worst complexity |
|---|---|---|
| 4−8 | $\Theta(1)$ | $\Theta(1)$ |
| 10−14 | $\Theta(1)$ | $\Theta(\|U\|\|D_j\|)$ |
| 17−24 | $\Theta(1)$ | $\Theta(\|D_j\|^2)$ |
| 25−29 | $\Theta(1)$ | $\Theta(1)$ |
| Total | $\Theta(q)$ | $\Theta(\|U\|^2)$ |

Algorithm 3 totally occurs three parts. Firstly, in step 2, we calculate the multi-threshold tolerance class the of inserted object $x_0$. The steps 4–11 of Algorithm 3 introduce procedures with regard to updated approximations of decision class $D_j$, while the inserted object $x_0$ is one element of $D_j$, where steps 5–10 compute the updated lower approximation and step 11 computes the updated upper approximation by Proposition 4.3. Secondly, in steps 12–24, Algorithm 3 describes processes concerning updated lower and upper approximation of decision class $D_j$, while the inserted object $x_0$ is not the element of $D_j$ and its decision value is equal to decision value of an object, which is one of $(U - D_j)$ objects, where steps 12–19 calculate the updated lower approximation and steps 19–24 compute the updated upper approximation by Proposition 4.4. Thirdly, in steps 26–32, Algorithm 3 investigates processes concerning updated approximations of decision class $D_j$, while decision value of the inserted object $x_0$ does not equal to any decision values of the universe $U$, where steps 27–31 count the updated lower approximation and step 32 computes the updated upper approximation by Proposition 4.5. Analogously, the flowchart of Algorithm 3 is shown in Fig. 2. Table 4 reveals the time complexity of Algorithm 3 under the best case (The Best Complexity) and the worst case (The Worst Complexity).

**Algorithm 3**: The dynamic algorithm for computing approximations after adding an object in IIDIS

**Input** : (1)the initial system $IIDIS=(U, C\cup\{d\}, V, f)$, where $B\subseteq C$.
(2)the initial multi-threshold tolerance classes $[x]_{R_B^{\alpha\beta\gamma}}$ $\forall x\in U$ and initial quotient set $U/d=\{D_1, D_2, \cdots, D_q\}$.
(3)the initial lower and upper approximations of $D_j$: $\underline{R_B^{\alpha\beta\gamma}}(D_j)$, $\overline{R_B^{\alpha\beta\gamma}}(D_j)(j=1,2,\cdots,q)$.
(4)the inserted object: $x_0$.

**Output** : the approximations of decision class $D_j$ after adding $x_0$: $\underline{R_B^{\alpha\beta\gamma}}(D_j')$, $\overline{R_B^{\alpha\beta\gamma}}(D_j')$.

1 **begin**
2 compute multi-threshold tolerance class of $x_0$: $[x_0]_{R_B^{\alpha\beta\gamma}}$;
3 **for** *j=1:q* **do**
4 **if** $x_0\in D_j$ **then**
5 $D_j'=D_j\cup\{x_0\}$;
6 **if** $[x_0]_{R_B^{\alpha\beta\gamma}}\subseteq D_j'$ **then**
7 $\underline{R_B^{\alpha\beta\gamma}}(D_j')=\underline{R_B^{\alpha\beta\gamma}}(D_j)\cup\{x_0\}$;
8 **else**
9 $\underline{R_B^{\alpha\beta\gamma}}(D_j')=\underline{R_B^{\alpha\beta\gamma}}(D_j)$;
10 **end**
11 $\overline{R_B^{\alpha\beta\gamma}}(D_j')=\overline{R_B^{\alpha\beta\gamma}}(D_j)\cup[x_0]_{R_B^{\alpha\beta\gamma}}$; /* compute the updated lower/upper approximation by Proposition 4.3 */
12 **else if** $\exists x\in(U-D_j), f(x,d)=f(x_0,d)$ **then**
13 $D_j'=D_j$;
14 let $\underline{R_B^{\alpha\beta\gamma}}(D_j')=\underline{R_B^{\alpha\beta\gamma}}(D_j)$;
15 **for** $x\in\underline{R_B^{\alpha\beta\gamma}}(D_j)$ **do**
16 **if** $x_0\in[x]'_{R_B^{\alpha\beta\gamma}}$ **then**
17 $\underline{R_B^{\alpha\beta\gamma}}(D_j')=\underline{R_B^{\alpha\beta\gamma}}(D_j')-\{x\}$;
18 **end**
19 **end**
20 **if** $\exists x\in D_j, x_0\in[x]_{R_B^{\alpha\beta\gamma}}$ **then**
21 $\overline{R_B^{\alpha\beta\gamma}}(D_j')=\overline{R_B^{\alpha\beta\gamma}}(D_j)\cup\{x_0\}$;
22 **else**
23 $\overline{R_B^{\alpha\beta\gamma}}(D_j')=\overline{R_B^{\alpha\beta\gamma}}(D_j)$; /* compute the updated lower/upper approximation of $D_j$ by Proposition 4.4 */
24 **end**
25 **else**
26 generate a novel decision class: $D_{q+1}$;
27 **if** $[x_0]_{R_B^{\alpha\beta\gamma}}\subseteq D_{q+1}$ **then**
28 $\underline{R_B^{\alpha\beta\gamma}}(D_{q+1})=\{x_0\}$;
29 **else**
30 $\underline{R_B^{\alpha\beta\gamma}}(D_{q+1})=\emptyset$;
31 **end**
32 $\overline{R_B^{\alpha\beta\gamma}}(D_{q+1})=[x_0]_{R_B^{\alpha\beta\gamma}}$; /* compute the updated lower/upper approximation of $D_j$ by Proposition 4.5 */
33 **end**
34 **end**
**return** : $\underline{R_B^{\alpha\beta\gamma}}(D_j')$, $\overline{R_B^{\alpha\beta\gamma}}(D_j')$.
35 **end**

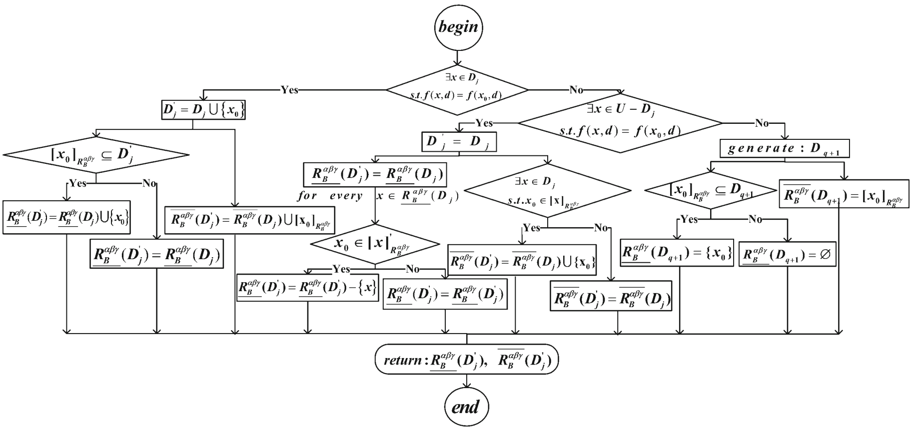


**Fig. 2** The flowchart of Algorithm 3

**Table 4** The time complexity of Algorithm 3

| Lines | The best complexity | The worst complexity |
|---|---|---|
| 2 | $\Theta(\|B\|\|U\|)$ | $\Theta(\|B\|\|U\|)$ |
| 5−11 | $\Theta(1)$ | $\Theta(\|D'_j\|)$ |
| 13−24 | $\Theta(1)$ | $\Theta(\|D_j\|)$ |
| 26−31 | $\Theta(1)$ | $\Theta(1)$ |
| Total | $\Theta(\|B\|\|U\|)$ | $\Theta(\|B\|\|U\|)$ |

### 5.4 Scalability analysis

Both the static and dynamic algorithms have good scalability. They can be easily extended to any other types of binary relations, such as similarity relation, interval-valued dominance relation [12,16], and so on. In other words, we provide a general algorithm framework on such binary relations. However, our static algorithm cannot be used for incremental learning. If we need to change objects of original data set, the static algorithm must restart. Obviously, the computational complexity is much higher when massive objects add to or delete from the initial data, but attribute set remains unchanged in IIDIS. In contrast, the dynamic algorithms are very flexible to undertake the above case. Because there is no necessary to reorganize the original data set. The idea of incremental learning is fully absorbed into the dynamic version. In real applications, the static algorithm is first used to generate the lower and upper approximations. Then, the dynamic algorithms are merged in incremental process, which are very effective.

## 6 Experimental analysis

Compared with the static algorithm, several experiments are proposed for verifying the efficiency and performance of the dynamic algorithms in the light of six data sets from UCI database in this section, namely “User Knowledge Modeling,” “Blood Transfusion Service Center,” “Wine Quality—Red,” “Letter Recognition (randomly selecting 3400 objects),”

**Table 5** The testing data sets

| Data sets | Abbreviation | Object | Attribute | Decision class |
|---|---|---|---|---|
| User Knowledge Modeling | UKM | 403 | 6 | 4 |
| Blood Transfusion Service Center | BTSC | 748 | 5 | 2 |
| Wine Quality-Red | WQR | 1599 | 12 | 6 |
| Letter Recognition | LR | 3400 | 16 | 26 |
| Wine Quality-White | WQW | 4898 | 12 | 7 |
| Pen-Based Recognition of Handwritten Digits | PBRHD | 10,992 | 17 | 10 |

"Wine Quality—White," and "Pen-Based Recognition of Handwritten Digits," which are outlined in Table 5. The testing results are running on personal computer with processor (2.7 GHz Intel Core i5) and memory (8 GB 1867 MHz DDR3). The platform of algorithms is MATLAB 2016B.

In fact, the attribute values of six data sets are real numbers. But, what we are investigating is IIDIS. So we need utilizing multiply error precision $\xi$ and missing rate $\pi$ ($\pi \in (0, 1)$) to process the data and change the data from real numbers to interval numbers. Let $DIS = (U, C \cup \{d\}, V, f)$ be a decision information system. All attribute values are single-valued. For any $x_i \in U$, $a_j \in C$, the attribute value of $x_i$ under the attribute $a_j$ can be written as $t = f(x_i, a_j)$. Firstly, we randomly choose $\lfloor \pi \times |U| \times |C| \rfloor$($\lfloor \cdot \rfloor$ is the meaning of taking an integer down) attribute values and turn them into missing values in order to construct an incomplete information system. These missing values are written as *. In this experiment, we let $\pi = 0.3$, $\xi = 0.05$, $\lambda = 0.5$, $\alpha = 0.6$, $\beta = 0.2$, $\gamma = 0.4$. But, the attribute value of $x_i$ under the decision attribute $d$ remains unchanged. Secondly, the interval number can be obtained by formula $t' = [(1 - \xi) \times t, (1 + \xi) \times t]$. In the process of whole experiments, we only study objects vary with time, but attribute set remains unchanged. In summary, an IIDIS is gained by this way.

To observe evidently gap of computation time between the static algorithm and the dynamic algorithms, we first let a data set constitutes initial system IIDIS and another data set(after adding objects to the universe or deleting objects from the universe) is viewed as testing system *IIDIS*$'$. The detailed description about variation of universe is as follows.

- We set the universe of IIDIS as initial object set $U$. Randomly selecting $\lfloor Rr \times |U| \rfloor$ objects are viewed as testing object set $U'$ in every experiment, where the removal ratio(Rr) is from 0.1 to 0.9.
- We let a half objects of the universe in IIDIS as initial object set $U$. A testing object set $U'$ is formed by adding $\lfloor Ir \times |U| \rfloor$ objects to the initial object set $U$ in every experiment, where the inserted ratio(Ir) is from 0.1 to 0.9.

In order to avoid the contingency of the experiment and reduce errors, we select a decision class and run five times for each test and then take the average of the five results as the final result. The results(the unit is second) are shown in Table 6 about time of computing lower and upper approximations of both Algorithm 1 and Algorithm 2, and the curve graph for every data set is portrayed in Fig. 3. Where the x-coordinate and y-coordinate represent the removal ratio, computation time of calculating lower and upper approximations, respectively.

In each sub-figure (1–6) of Fig. 3, we can observe its line trend in totally. The computation time of static algorithm is reducing largely with the increasement of the removal ratio. And the computation time of dynamic algorithm is decreasing smoothly along with the increasement

**Table 6** The computation time of both Algorithm 1 and Algorithm 2 with different removal ratios

| The removal ratio | UKM | | BTSC | | WQR | | LR | | WQW | | PBRHD | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Static | Dynamic | Static | Dynamic | Static | Dynamic | Static | Dynamic | Static | Dynamic | Static | Dynamic |
| 0.1 | 0.600 | 0.100 | 2.260 | 0.260 | 2.132 | 0.134 | 54.470 | 1.100 | 21.502 | 0.318 | 614.220 | 4.110 |
| 0.2 | 0.460 | 0.100 | 1.750 | 0.240 | 1.638 | 0.090 | 42.310 | 0.970 | 16.016 | 0.286 | 447.610 | 3.600 |
| 0.3 | 0.340 | 0.090 | 1.210 | 0.210 | 1.488 | 0.122 | 33.420 | 0.890 | 12.118 | 0.244 | 363.630 | 5.380 |
| 0.4 | 0.280 | 0.070 | 0.910 | 0.180 | 1.020 | 0.066 | 23.940 | 0.860 | 8.886 | 0.214 | 263.080 | 4.530 |
| 0.5 | 0.160 | 0.050 | 0.630 | 0.160 | 0.732 | 0.058 | 16.480 | 0.670 | 6.150 | 0.178 | 164.910 | 3.550 |
| 0.6 | 0.120 | 0.040 | 0.420 | 0.120 | 0.434 | 0.048 | 11.090 | 0.500 | 3.908 | 0.138 | 104.390 | 2.870 |
| 0.7 | 0.60 | 0.020 | 0.250 | 0.080 | 0.244 | 0.032 | 6.000 | 0.390 | 2.292 | 0.218 | 58.550 | 2.200 |
| 0.8 | 0.040 | 0.020 | 0.120 | 0.070 | 0.118 | 0.030 | 2.840 | 0.230 | 1.012 | 0.142 | 26.160 | 1.480 |
| 0.9 | 0.000 | 0.010 | 0.040 | 0.040 | 0.034 | 0.016 | 0.700 | 0.150 | 0.252 | 0.064 | 6.450 | 0.760 |

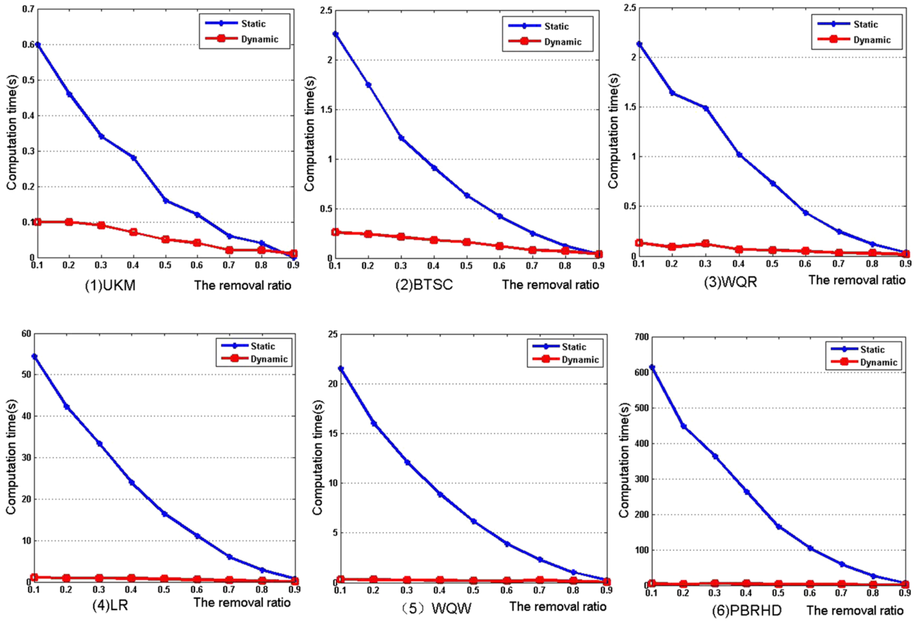


**Fig. 3** Comparison of running time of both Algorithms 1 and 2

of the removal ratio in principle. When the removal ratio reaches a certain value, the static algorithm has the same time with the dynamic algorithm in calculating approximations. In addition, it is obviously known that dynamic algorithm performs faster than static algorithm.

Similarly, the results(the unit is second) are exhibited in Table 7 about time of computing lower and upper approximations of both Algorithm 1 and Algorithm 3 and the curve graph for every data set is depicted in Fig. 4. The x-coordinate and y-coordinate indicate the inserted ratio, computation time of calculating lower and upper approximations, respectively.

In each sub-figure (1–6) of Fig. 4, we can observe its line trend in totally. The computation time of the static algorithm and dynamic algorithm is both increasing monotonically with the increasement of the inserted ratio. The curve graph of two algorithms has similar growth rates. But, it is distinctly shown that dynamic algorithm performs faster than static algorithm. Therefore, the dynamic algorithms are faster and more efficient than the static algorithm concerning time of calculating approximation sets in IIDIS whether objects are deleted from the universe or objects are inserted into the universe.

## 7 Conclusion

Information system may change with the variation of obtained data in real life. Approximate sets will also evolve over time simultaneously. How to reduce the time to calculate the approximate sets is an important content of discovering knowledge in RST. In this article, we establish a multi-threshold tolerance relation through set pair analysis theory and construct rough set model in the IIDIS that is a promotion of Pawlak information system. Then, the static algorithm is proposed by definition of approximations, which can directly acquire approximation sets step by step. The dynamic algorithms are presented to compute

**Table 7** The computation time of both Algorithm 1 and Algorithm 3 with different inserted ratios

| The inserted ratio | UKM | | BTSC | | WQR | | LR | | WQW | | PBRHD | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Static | Dynamic | Static | Dynamic | Static | Dynamic | Static | Dynamic | Static | Dynamic | Static | Dynamic |
| 0.1 | 0.570 | 0.050 | 1.270 | 0.200 | 0.992 | 0.172 | 20.710 | 3.680 | 7.616 | 1.358 | 220.370 | 38.080 |
| 0.2 | 0.560 | 0.090 | 1.380 | 0.330 | 1.012 | 0.292 | 25.020 | 7.500 | 9.048 | 2.802 | 260.120 | 79.910 |
| 0.3 | 0.620 | 0.120 | 1.470 | 0.450 | 1.402 | 0.538 | 28.800 | 11.760 | 10.434 | 4.344 | 310.080 | 126.350 |
| 0.4 | 0.660 | 0.170 | 1.690 | 0.620 | 1.356 | 0.692 | 33.500 | 17.140 | 12.104 | 6.022 | 360.860 | 174.760 |
| 0.5 | 0.690 | 0.210 | 1.910 | 0.790 | 1.640 | 0.868 | 38.370 | 20.960 | 13.848 | 7.786 | 371.780 | 206.360 |
| 0.6 | 0.780 | 0.290 | 2.000 | 0.960 | 1.900 | 1.062 | 43.740 | 26.360 | 15.732 | 9.734 | 419.070 | 255.480 |
| 0.7 | 0.810 | 0.340 | 2.340 | 1.210 | 2.078 | 1.320 | 48.240 | 31.810 | 18.914 | 12.844 | 473.570 | 309.460 |
| 0.8 | 0.950 | 0.410 | 2.480 | 1.580 | 2.372 | 1.546 | 54.420 | 36.860 | 20.634 | 14.694 | 526.050 | 366.430 |
| 0.9 | 0.980 | 0.490 | 2.870 | 1.640 | 2.646 | 1.842 | 61.380 | 44.440 | 22.744 | 16.642 | 586.610 | 429.080 |

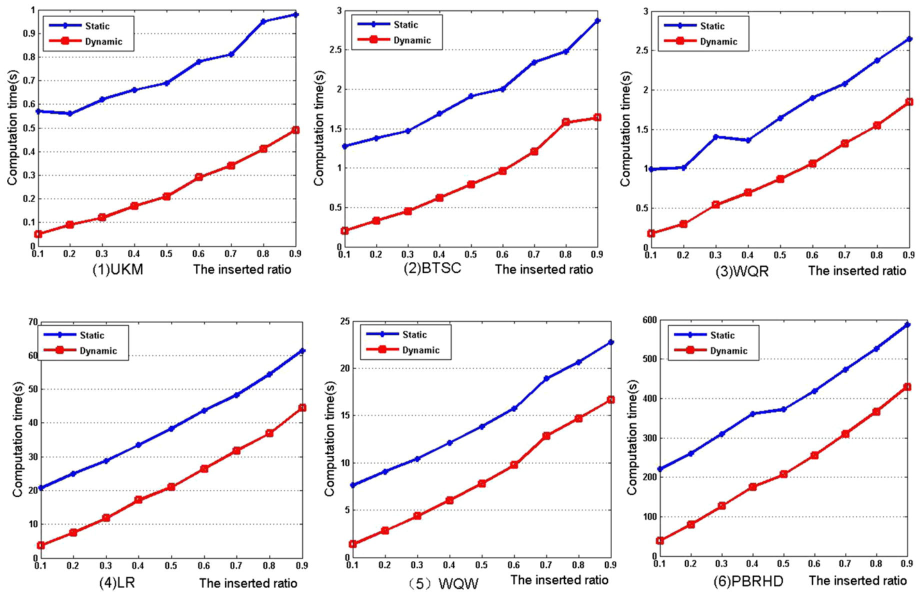


**Fig. 4** Comparison of running time of both Algorithms 1 and 3

the upper and lower approximation according to the previous data and knowledge and several propositions, which are exhibited when objects vary, but the attribute set remains unchanged. Finally, the experimental results make clear that the presented dynamic algorithms have less time than the static algorithm when calculating the approximate sets. In addition, this article only discusses the variation of objects.

Later, we will consider that a multi-threshold tolerance relation can be viewed as one granularity and further explore approaches to deal with the issue about updating approximations for multi-granulation rough set in IIDIS.

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